NON-NEWTONIAN EFFECTS IN VISCOUS FLOWS

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ABSTRACT

Revision of the mathematical formalism of fluid dynamics suggests that some physical inconsistencies (infinite time of approaching equilibria, and fully deterministic solutions to the Navier-Stokes equations) can be removed by relaxing the Lipschitz conditions, i.e., the boundedness of the derivatives, in the constitutive equations. Physically such a modification can be interpreted as an incorporation of an infinitesimal static friction in the constitutive law. A modified version of the Navier-Stokes equations is introduced, discussed, and illustrated by examples. It is demonstrated that all the new effects in the modified model emerge within vanishingly small neighborhoods of equilibrium states which are the only domains where the governing equations are different from classical.

1. INTRODUCTION

One of the central problems in fluid dynamics is to explain how motion which is described by fully deterministic governing equations can be random. Indeed, let us consider exponential growth of a vorticity component w:

$$\omega = \omega_o e^{\lambda t}, \quad 0 < \lambda < \infty \tag{1}$$

Obviously a solution with an infinitesimally close initial condition

$$\hat{\omega} = \omega_1 e^{\lambda t}, \; \omega_1 = \omega_o + \varepsilon, \; \varepsilon \to 0 \tag{2}$$

will remain infinitesimally close to the original one:

$$| \omega - \hat{\omega} | = \varepsilon e^{\lambda t} \to 0 \text{ at } t < \infty \text{ if } \varepsilon \to 0 \tag{3}$$

during all bounded time intervals.

This means that random solutions can result only from random initial conditions when ε in (2) is small, but finite rather than infinitesimal. in other words, classical fluid dynamics can explain amplificat ions of random motions 1 › y the mechanism of instability, but it cannot represent their origin using mathemat ical formalism.

The recent discovery of chaotic motions in nonlinear dynamics demonstrates that the same kind of problems exists in tile general formalism of Newtonian mechanics when motions described by fully deterministic models appear to be random. A revision of this formalism was presented by Zak [1- 3], and here we will briefly discuss it.

The governing equations of classical dynamics may be derived either from Lagrangian functions, from variational principles, or dime.tly from Newton's laws of motion, and they may be presented in various equivalent forms. However, three is one mathematical restriction on all such forms: the differential equations describing a dynamical system

$$\dot{x}_i = v_i(x_1, x_2, \cdots, x_n) \quad i = 1, 2, \cdots, n \tag{4}$$

must satisfy the Lipschitz condition, which expresses that all the derivatives

$$\left| \frac{\partial v_i}{\partial x_j} \right| < \infty \tag{5}$$

must be bounded. This mathematical restriction guarantees the uniqueness of the solution of (4), subject to fixed initial conditions.

This condition allows one to describe the Newtonian dynamics within the mathematid framework of the classical theory of differential equations which guarantees its predict ability. That, in t urn, leads to such effects as infinite time of approaching an attractor, infinite time for escape of a repeller if changes in initial condi tions are infinitesimal, untractability of two trajectories which originally are "very close", but diverge exponentially, etc.

Hence, there are a variety of phenomena whose explanations cannot be based directly upon classical dynamics: in addition, they require some "words" about a scale of observa-tion, "very close" trajectories, etc.,

Turning to the governing equations of classical dynamics:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = -\frac{\partial R}{\partial \dot{q}_i}, \qquad i = 1, 2, \cdots, n, \tag{6}$$

where L is the Langrangian, q_i, $\dot{q}_i$ are the generalized coordinates and velocities, and R is the dissipation function, one should recall that the structure of $R(\dot{q}_i, \cdots, \dot{q}_n)$ is not prescribed by Newton's laws. Some additional assumptions are to be made in order to define it. The "natural" assumption (which has never been challenged) is that these functions can be expanded in a Taylor series with respect to equilibrium states: $\dot{q}_i = 0$. obviously, this requires the existence of the derivative: $\left| \frac{\partial^2 R}{\partial \dot{q}_i \partial \dot{q}_j} \right| < \infty$ at $\dot{q}_i \to 0$.

A departure from that condition was proposed in [2], (see the Appendix), where the following dissipation function was introduced:

$$R = \frac{1}{k+1} \sum_i \alpha_i \left| \sum_j \frac{\partial r_i}{\partial q_j} \dot{q}_j \right|^{k+1} \tag{7}$$

in which α_i are positive constants, N is the number of the friction forces applied to the points r_i, and

$$k = \frac{p}{p+2} < 1, p \gg 1, \tag{8}$$

while p is a large odd number.

By selecting a large p, one can make k close to 1 so that ("7) is almost identical to the classical one (when k = 1.) everywhere, excluding a small neighborhood of the equilibrium point $\dot{q}_j = 0$, while at this point:

$$\left| \frac{\partial^2 R}{\partial \dot{q}_i \partial \dot{q}_j} \right| \to \infty \quad \text{at} \quad \dot{q}_j \to 0. \tag{9}$$

Hence, the Lipschitz condition is violated, the friction force $F_i = --\sim$ grows sharply at the equilibrium point, and then it gradually approaches its "classical" value. This effect can be interpreted as a mathematical representation of a jump from st atic to kinetic friction, when the dissipation force does not vanish with the velocity,

It appears that this "small " difference between friction forces at $k = 1$ and $k < 1$ leads to fundament al changes in Newtonian dynamics. In order to demonstrate it, we will consider the relationship between the total energy E and the dissipation function R:

$$\frac{dE}{dt} = -\sum_i \dot{q}_i \frac{\partial R}{\partial \dot{q}_i} = -(k+1)R. \tag{lo}$$

Within a small neighborhood of an equilibrium state (where the potential energy can be set zero), the energy E and the dissipation function R have the order, respectively:

$$E \sim \dot{q}_i^2, \qquad R \sim \dot{q}_i^{k+1} \qquad \text{at} \qquad E \to o. \tag{11}$$

Hence, the asymptotic form of (10) can be presented as:

$$\frac{dE}{dt} = AE^{k+1/2} \qquad \text{at} \qquad E \to 0, \qquad A = \text{const}. \tag{12}$$

If $A < 0$ and $k < 1$, the equilibrium state $E = O$ is an attractor where the Lipschitz condition $(/ \, d\dot{E}/dE \,| \to \infty$ at $E \to O)$ is violated. Such a terminal [1] attractor is approached by the solution originated at $E = \triangle E_o > 0$, in finite time:

$$t_0 = \int_{\triangle E_o}^{0} \frac{dE}{AE^{(k+1/2)}} = \frac{2 \triangle E_0^{(1-k/2)}}{(1--k) \mid A \mid} < \infty. \tag{13}$$

Obviously, this integral diverges in the classical case $k \geq 1$, where $t_o \to \infty$. The motion described by (12) has a singular solution $E \equiv O$ and a regular solution:

$$E == [\triangle E_0^{(1-k/2)} \text{-I-} \frac{1}{2} A(1-k)t]^{(2/1--k)} \tag{14}$$

In a finite time, the motion can reach the equilibrium and switch to the singular solution $E \equiv O$, and this switch is irreversible.

The property of the solution to the equation $\dot{x} = -x^{1/3}$ (which is of the type of Eq. (12)), is illustrated by Fig. 1,a.

As is well known from dynamics of nonconservative systems, dissipative forces can destabilize the motion when they feed the external energy into the system (the transmission of energy from laminar to turbulent flow in fluid dynamics, or from rotations to oscillations in the dynamics of flexible systems). In terms of (1 2), it would mean that $A > 0$, and the equilibrium state $E = 0$ becomes a terminal repeller[1].

If the initial condition is infinitely close to this repeller, the transient solution will escape it during a finite time period (Fig. 1),

$$t_o = \int_{\varepsilon \to 0}^{\Delta E_o} \frac{dE}{AE^{\left(\frac{k+1}{2}\right)}} = \frac{2 \Delta E_o^{(1-k/2)}}{(1-k)A} = < \infty, \tag{15}$$

while for a regular repeller, the tire? would be infinite.

Expressing (12) in terms of the? velocity at $i = 1$, $\dot{q}_1 = v$,

$$\dot{v} = Bv^k, \qquad B = \text{const} > 0, \tag{16}$$

one arrives at the following solution:

$$v = \pm \left\{ [B(1 - k)t]^{p+2} \right\}^{1/2}. \tag{17}$$

As in the case of a terminal attractor, here the motion is also irreversible: the time-backward motion obtained by formal time reversal $t \to -t$ in (17) is imaginary, since p is an odd number (see (8)).

But in addition to that, the terminal repellers possess even more surprising characteristics: the solution (1 7) becomes totally unpredictable. Indeed, two different motions described by the solution (1 7) are possible for "almost the same" ($v_o = +\varepsilon + 0$, or $v_o = -\varepsilon + 0$ at $t \to 0$) initial conditions. Assuming that positive and negative disturbane.es $\pm\varepsilon$ occur with equal probability 0.5, one arrives at the situation when instability and non uniqueness of the solution impart elements of stochasticity into the postinstability behavior.

Thus, a terminal repeller represents a vanishingly short, but infinitely powerful "pulse of unpredictability" which is pumped into the system via terminal dissipative forces. Obviously, failure of the uniqueness of the solution here results from the violation of the Lipschitz condition at $v = 0$.

As is known from classical dynamics, combination of stabilizing and destabilizing effects can lead to chaos. In order to describe similar effects in dynamics with terminal dissipative forces, let us slightly modify (16) assuming that $B = B_o \cos wt$.

Then stabilization and destabilization effects alternate. With the initial condition $v \to 0$ at $t \to 0$, the exact solution to (16) consists of both a regular solution:

$$v = \pm \left[\frac{B_o(1-k)}{w} \sin \omega t \right]^{(1/1-k)}, \quad v \neq 0, \tag{18}$$

and a singular solution $v = 0$. During the first period $0 < t < \pi/2\omega$, the? equilibrium point $v = 0$ is a terminal repeller. Therefore, within this interval, the motion can follow one of two possible trajectories (18) (each with probability 1/2). during the next period $\pi/2\omega < t < 3\pi/2\omega$, the equilibrium point becomes a terminal attractor; the solution approaches it at $t = \pi\omega$ and it remains motionless until $t > 3\pi/2\omega$. After that the. terminal attractor converts into a terminal repeller, and the solution escapes again, etc. It is important to notice that each time the system escapes the terminal repeller, the solution splits into two symmetric branches, so that there are 2^n possible scenarios of oscillations with respect to the center $v = 0$, while each scenario has the probability 2^{-n} (n is the number of cycles), Hence, the motion (18) resembles chaotic oscillations known from classical dynamics. It combines random characteristics with the attraction to a center. However, in the classical case, the chaos is caused by a supersensitivity to the initial Conditions, while the uniqueness of the solution for fixed initial conditions is guaranteed. In contrast to that, the chaos in the oscillations (18) is caused by the failure of the uniqueness of the solution at the equilibrium points, and it has a well-organized probabilistic structure. Since tile time of approaching the equilibrium point $v = 0$ by the solution (18) is finite, this type of chaos can be called terminal [1-3] or nondeterministic.

Within the framework of terminal dynamics, formations of new patterns of motion can be understood as chains of terminal attractions and repulsions. As demonstrated above, during each terminal repulsion the solution splits into two symmetric branches, and the motion can follow each of them with equal probability.

As shown in [2,3] such a scenario can be described by a system of differential equations with terminal equilibrium points. In contradistinction to stochastic equations, here randomness results from the violat ion of the uniqueness of t he so lut ions, and therefore, the differential operator itself generates random motions. Because of that, terminal dynamics possessed a well-organized probabilistic structure, described by a Fokker-Plane.k type of equation whose coefficients are uniquely defined by fully det ern inistic parameters of the original dynamical system [2,3]. At the same time, it should be stressed again that all the new effects of termin al dynamics emerge within vanishingly small neighborhoods of equilibrium st ates which are the only domains where the govern ing equations are different from classical.

Now we will formulate the basic physical assumptions un derlying the paper.

The dynamics of a fluid, on the macroscopic level, is described by the Navier-Stokes equations which are based upon Newton>s laws. H owever, besides that, some additional physical assumptions are needed to introduce the dissipation function which defines the rheology of the stress-strain relationships in a fluid. On the macrc)sc.epic level, these assumptions are based upon the two laws of t hermod ynami cs, as well as upon the principles of kinetics. The rest of the "details" must be found from experiments. However, there is another set of assumptions (which are of a mathema t ical nat ure) used in formulation of the Navier-Stokes equations. The most powerful of them is the requirement of differentiability (as many times as necessary) of all the macrosc opic parameters with respect to time and space coordinates. Such a requirement is fully compatible with the principles of the macroscopic level of description. However, another mathematical assumption about the expandability of the dissipation function in a Taylor series with respect to the state of Equilibrium (which is used for deriving tilt? simplest version of the constitutive law) is not so "innocent'' as it may look on first sight. Indeed, from the physical viewpoint, it eliminates . the possibility of static friction or plasticity effects which may exist within the infinitely small neighborhood of equilibrium states. The models which describe such effects are well known [5], and they are fully compatible with the laws of mechanics and thermodynamics.. From the mathematical viewpoint, the assumption ı about the expandability y of a Taylor series of the dissipation function enforces the Lipschitz condition at the equilibrium states, and that, in turn, leads to infinite time of approaching these states. The main objective of this paper is to show that by relaxing the Lipschitz condition in the constitutive law of viscous liquids one will 1 ıave a much more realistic scenario of behavior of liquids in the domains approaching to and departing from the equilibrium states.

2. CONSTITUTIVE EQUATIONS

Following the ideas described in the Introduction, we will introduce and discuss ıere the non- Lipschitzian version of the dissipation function for a liquid in the same way as it was done in (7).

As follows from extremum principles in irreversible thermodynamics[4], the simplest form of the dissipation function for an isotropic liquid which may incorporate non Lipschitzian properties, is the following:

$$R = \mathcal{D}(I_2) \tag{19}$$

where $\mathcal{D}$ is a positive-definite differentiable function of the second invariant I_2 of the rate-of-strain, tensor ε:

$$\varepsilon = \mathrm{def}\ \mathbf{v} = \frac{1}{2}(\nabla\mathbf{v}\ -1\ \nabla\mathbf{v}^T), \tag{20}$$

i.e.,

$$I_2 = \frac{1}{2}\varepsilon_{jk}\varepsilon_{kj}. \tag{21}$$

Here ε_{jk} are the components of the tensor ε:

$$\varepsilon_{jk} = \frac{1}{2}\left(\frac{\partial v_j}{\partial x_k} + \frac{\partial v_k}{\partial x_j}\right) \tag{22}$$

while v_j are the components of the velocity vector v.

The dissipation function (19) defines the deviatoric stress tensor:

$$\sigma_{jk} = \frac{1}{2}\frac{\partial \mathcal{D}}{\partial I_2}\varepsilon_{jk} \tag{23}$$

The isotropic, part of the stress tensor can be presented in the simplest form (since $\nabla.\mathbf{v} = 0$):

$$\overset{*}{\sigma}_{ii} = -p \tag{24}$$

where p is the pressure.

Turning back to Eq. (19) let us specify the dissipation function as following:

$$\rho\mathcal{D} = 4\mu'\left(\frac{\sqrt{I_2}}{\varepsilon_o^*}\right)^{k-1} I_2 \tag{25}$$

where μ' and ε_o are positive constants with the dimensions of viscosity μ and the rate-of-strain ε, respectively, while k <1 is given by Eq. (8).

Then the deviatoric stress tensor follows from Eqs. (23) and (25):

$$\sigma_{jk} = 2\mu'\left(\frac{\sqrt{I_2}}{\varepsilon_o}\right)^{k-1} \varepsilon_{jk} \tag{26}$$

Eq. (26) is different from the Newtonian liquid only within an infinitely small neighborhood of the equilibria states where

$$I_2 \to 0, \qquad i.e., \varepsilon_{jk}, \sigma_{jk} \to 0 \tag{27}$$

Otherwise

$$\left(\frac{\sqrt{I_2}}{\varepsilon_o}\right)^{k-1} \simeq 1, \tag{28}$$

as follows from (26) and (2 S), $\mu' = 2\mu$ where μ is the c.lassie.al viscosity.

One can verify that the Lipschitz condition for the function (26) at $I_2 + 0$ is violated since

$$\left| \frac{\partial\sigma_{jk}}{\partial\varepsilon_{jk}} \right| \to \infty \qquad \text{at} \qquad I_2 \to 0 \tag{29}$$

Mathematical consequences of this property (which are similar to those described in the Introduction) will be discussed in the next section.

The physical meaning of the property (23) is clear: it describes a limit case of a viscoplastic body when the domain of plasticit y is vanishing y small, see Fig. 2.

Let US summarize now all the? arguments for selecting k based upon Eq. (8).

Firstly, k must be dose. to 1

$$| \; k - 1 \; | \ll 1$$

to preserve classical results in domains which exclude only small neighborhoods around equilibria (see Eq. (28)).

Secondly, k must be less then 1

$$0 < k < 1$$

to introduce the plasticity effects around equilibria via the relaxation of the Lipschitz condition (see Eq. (29)).

Thirdly, k must be represented by a fraction with an odd numerator and an odd denominator in order to preserve the stress-strain relationships in the form given in Fig. 2. Indeed, in case of an even numerator, the left branch in Fig. 2 will be positive, while in case of an even denominator, it will be imaginary. Obviously both cases are physically unrealist ic.

Hence, actually Eq. (8) minimizes the degree of arbitrariness to which the constant k is defined. It should be noticed that similar model was discussed by H. Ziegler [5] where he introduced a limit case of a viscoplastic model.

In case of a two-dimensional flow where the velocity can be expressed via the stream function ψ:

$$v_1 = \frac{\partial \psi}{\partial x_2}, \qquad v_2 = -\frac{\partial \psi}{\partial x_1} \tag{30}$$

and therefore,

$$I_2 = \frac{1}{2}(\varepsilon_{11}^2 + \varepsilon_{22}^2 + 2\varepsilon_{12}^2) = \left(\frac{\partial^2\psi}{\partial x_1 \partial x_2}\right)^2 + \left(\frac{\partial^2\psi}{\partial x_2^2} - \frac{\partial^2\psi}{\partial x_1^2}\right)^2 \tag{31}$$

Eq. (26) reads:

$$\sigma_{11} = \mu \varepsilon_o^{1-k} \left[\left(\frac{\partial^2\psi}{\partial x_1 \partial x_2}\right)^2 + \left(\frac{\partial^2\psi}{\partial x_2^2} - \frac{\partial^2\psi}{\partial x_1^2}\right)^2\right]^{\frac{k-1}{2}} \frac{\partial^2\psi}{\partial x_1 \partial x_2} \tag{32}$$

$$u'' = -\mu \,\&_o^{1-k} \left[\left(\frac{\partial^2\psi}{\partial x_1 \partial x_2}\right)^2 + \left(\frac{\partial^2\psi}{\partial x_2^2} - \frac{\partial^2\psi}{\partial x_1^2}\right)^2\right]^{\frac{k-1}{2}} \frac{\partial^2\psi}{\partial x_1 \partial x_2} \tag{33}$$

$$\sigma_{12} = -\mu \varepsilon_o^{1-k} \left[\left(\frac{\partial^2\psi}{\partial x_1 \partial x_2}\right)^2 + \left(\frac{\partial^2\psi}{\partial x_2^2} - \frac{\partial^2\psi}{\partial x_1^2}\right)^2\right]^{\frac{k-1}{2}} \left(\frac{\partial^2\psi}{\partial x_2^2} - \frac{\partial^2\psi}{\partial x_1^2}\right) \tag{34}$$

In the simplest case of a two-dimensional unidirectional flow:

$$v_1 = u, \; v_2 = v_3 = 0, \quad \sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{13} = \sigma_{23} = 0 \tag{35}$$

the only non-zero component of the stress tensor is:

$$\sigma_{12} = \mu' \varepsilon_o^{1-k} \left(\frac{\partial u}{\partial x_2}\right)^k, \quad \mu' = 2\mu \tag{36}$$

This case was analyzed in [2].

Eq. (31)-(36) will be exploited in our further discussions.

3. GOVERNING EQUATIONS

Substituting the constitutive equations (26) into the momentum equations:

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \nabla \mathbf{v}\right) = -\nabla p + \nabla . \sigma \tag{37}$$

and taking into account the identity

$$\nabla.(\alpha A) = \alpha \nabla .A + A. \nabla \alpha \tag{38}$$

holding for arbitrary tensor A and scalar α, one obtains the non-Lipschitzian version of the Navier-Stokes equation:

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \nabla \mathbf{v} \right) = - \nabla p + \mu^* [I_2^{\frac{k-1}{2}} \nabla .(\nabla \mathbf{v} + \nabla \mathbf{v}^T) + (\nabla \mathbf{v} + \nabla \mathbf{v}^T). \nabla (I_2^{\frac{k-1}{2}})] \tag{39}$$

where

$$\mu^* = \frac{1}{2} \mu' \varepsilon_o^{1-k} = \text{const}$$

This equation must be complemented by the condition of incompressibility:

$$\nabla.\mathbf{v} = 0 \tag{40}$$

Eq. (39) is different from the Navier-Stokes equation only within vanishingly small neighborhoods of equilibria whine

$$I_2 + 0, \text{ i.e. } \sigma \rightarrow 0, \ \varepsilon \rightarrow 0. \tag{41}$$

Otherwise

$$I_2^{\frac{k-1}{2}} \simeq 1, \ \nabla(I_2^{\frac{k-1}{2}}) \simeq 0, \ \varepsilon^{1-k} \simeq 1 \tag{42}$$

which reduces Eq. (39) to its classical form:

$$\rho(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \nabla \mathbf{v}) = - \nabla p + \mu \nabla .(\nabla \mathbf{v} + \nabla \mathbf{v}^T) \tag{43}$$

In the particular case (35)-(36) of a two-dimensional unidirectional flow, Eqs. (39)-*(40)* reduce to one equation:

$$\frac{\partial u}{\partial t} = \nu^*\left(\frac{\partial u}{\partial x_2}\right)^{k-1} \frac{\partial^2 u}{\partial x_2^2}, \quad \nu = \frac{\mu}{\rho} \tag{44}$$

which is different from the classical diffusion equation

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x_2^2} \tag{45}$$

only if

$$\left| \frac{\partial u}{\partial x} \right| \to 0 \tag{46}$$

4. LARGE-SCALE EFFECTS

Let us evaluate the range of motion scales where the proposed model describes special effects missed in the classical description.

Turning to the constitutive equation (19) and expanding it in a Taylor series:

$$R = a_1 \; I_2 + a_2 I_2^2 + \cdots \quad \text{etc.} \tag{47}$$

one can verify that the Newtonian liquid described by the Navier- Stekes equation corresponds to the case when only the first term in Eq. (47) is kept. That is why this simplest model is valid only for such velocity gradients which are relatively small in comparison to those on the. molecular scale.

The same conclusion can be made based upon statistical mechanical concepts when the non- equilibrium component of the Maxwell distribution function is expanded in a Taylor series.

However, there is another possibility in representing Eq. (47), for instance:

$$R = a_{-1}(I_2)^{-1} + a_{-2}(I_2)^{-2} + \ldots \text{etc.} \tag{48}$$

It has never been exploited because of the mathematical "inconvenience" caused by the singularity y at equilibria where $I_2 \rightarrow 0$.

The proposed model defined by Eq. (25) belongs to the same type as Eq. (48), although it has a weaker singularity:

$$R \rightarrow 0, \quad \text{but} \quad \frac{dR}{dI_2} \rightarrow \infty \quad \text{at } I_2 \rightarrow 0 \tag{49}$$

i.e., at equilibria the Lipschitz condition is violated.

It should be expected that contrary to the case (47), the constitutive laws of the type. (48), including the proposal model, are taking into account the large scale motion effects. Indeed, as pointed out above, the proposed model describes new effects when the? velocity gradients are small in the sense that

$$\sqrt{I_2} \leq \varepsilon_0 \tag{50}$$

Here ε_0 is the physical constant of the liquid introduced by the constitutive equation (25). Since its dimensionality is:

$$[\varepsilon_0] = \frac{1}{\text{sec}}, \tag{51}$$

one can int reduce the time scale T_0 of the mot ions descril)ed by the proposed model. Indeed, based upon Eqs. (50) and (51), one obtains:

$$\sqrt{I_2} \sim \frac{v_0}{L_0} \sim \frac{1}{T_0} \leq \varepsilon_0 \tag{52}$$

whence

$$T_0 \geq \frac{1}{\varepsilon_0} \tag{53}$$

The length scale L can be found from the condition:

$$L_0 = \sqrt{\nu T_0} \geq \sqrt{\frac{\nu}{\varepsilon_0}}, \tag{54}$$

The evaluations *(53)* and *(54)* demonstrate that the proposed model describes large scale motion effects, i.e., motions close to equilibria where the velocities and their gradients are datively small.

Turning to the governing equations (39), one can simplify them by ignoring the convection terms of the acceleration which are small in comparison to similar viscous terms in the domain of large scale motions (53), (54):

$$\rho\frac{\partial \mathbf{v}}{\partial t} = -\nabla p + \mu^* \left[I_2^{\frac{k-1}{2}} \nabla .(\nabla \mathbf{v} + \nabla \mathbf{v}^T) + (\nabla \mathbf{v} + \nabla \mathbf{v}^T). \nabla (I_2^{\frac{k-1}{2}}) \right] \tag{55}$$

The expression for the energy dissipation:

$$\dot{E}_{\mathcal{D}} = -\frac{\nu^*}{2}\int_v \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)^{k+1} dv \tag{56}$$

does not differ much from the? classical case (k = 1), which means that it decreases with the grows of the length scale:

$$\dot{E}_{\mathcal{D}} \sim O\left(L_0^{-\frac{1}{2}} \nu_0^{\frac{1}{2}} R_e^{\frac{1}{2}}\right) \tag{57}$$

The same can be said about the dissipation stresses (see Eq. 26).

However, the dissipation forces $\nabla.\sigma$, i.e., the contribution of the dissipation stresses to the momentum equation, differ significantly from the classical case k = 1: they grow sharply with t he decrease of the velocit y gradients, becoming unbounded at the equilibrium. As will be shown below, the last property is responsible for a finite time of approaching equilibria. From a physical viewpoint this means that at equilibria the dissipation is carried out, by static. friction.

Thus, the modification of the constitutive law which relaxes the Lipschitz condition at equilibria by introducing a vanishingly small static friction, eliminates one of the least '(damaging') inconsistency in fluid dynamics (as well as in classical dynamics): theoretically infinite time of approaching equilibria. However, as a "side-effect", it eliminates a more "damaging" inconsistency: the occurrence of stochastic motions in flows which are described by fully deterministic hydrodynamical models. As will be shown below, the relaxation of the Lipschitz condition at equilibria in combination with instability may cause the failure of the uniqueness of solution to Eqs. (39) and (40), and that can be represented by additional stochastic components in the. solution. The instability mentioned above is of the same type? as in Eq. (16): it is a supersensitivity to infinitesimal changes of initial condition. At first sight it seems unlikely that at equilibria where the actual viscosity is very large (strictly speaking, it is a static friction rather than viscosity), any instability can occur at all. However, as well-known from the theory of hydrodynamic stability, viscosity can be a destabilizing factor, for instance, in parallel flows where the conditions (41) are well satisfied).

5. BEHAVIOR AROUND EQUILIBRIA

In this section we will analyze the behavior of a non-Lipschitzian liquid within vanishingly small neighborhoods of equilibrium states where the condition (41) holds.

Our analysis will be based upon the energy balance for the liquid in a volume v with the boundary s which for any isotropic liquid can be presented in the following form [1]:

$$\frac{\partial}{\partial t}\int_{\mathrm{v}}\frac{\rho v^2}{2}d\mathrm{v} = -\oint_s\left[\rho\mathbf{v}\left(\frac{v^2}{2}+\frac{p}{\rho}\right)-(\mathbf{v}\sigma)\right]d\mathbf{n}-\int_{\mathrm{v}}\sigma_{ik}\frac{\partial v_i}{\partial x_k}d\mathrm{v} \tag{58}$$

Here $\mathbf{v}\sigma$ denotes a vector with the components $v_i\sigma_{ik}$, and $\mathbf{n}$ is the unit normal to the surface s.

Confining our discussion to a two-dimensional flow and utilizing the expressions given by Eqs. (30)-(34), one rewrites Eq. (59) in terms of the stream function ψ:

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathrm{v}}\left[\left(\frac{\partial\psi}{\partial x_1}\right)^2+\left(\frac{\partial\psi}{\partial x_2}\right)^2\right]dxdy = -\frac{\nu^*}{2}\int_{\mathrm{v}}\left(\frac{\partial^2\psi}{\partial x_2^2}-\frac{\partial^2\psi}{\partial x_1^2}\right)^{k+1}dxdy$$

$$+2\nu^*\oint\left\{\left[\left(\frac{\partial^2\psi}{\partial x_1\partial x_2}\right)^2+\left(\frac{\partial^2\psi}{\partial x_2^2}-\frac{\partial^2\psi}{\partial x_1^2}\right)^2\right]^{\frac{k-1}{2}}\left[\frac{\partial^2\psi}{\partial x_1\partial x_2}\frac{\partial\psi}{\partial x_2}+\left(\frac{\partial^2\psi}{\partial x_1^2}-\frac{\partial^2\psi}{\partial x_2^2}\right)\frac{\partial\psi}{\partial x_1}\right]\cos\alpha_1\right.$$

$$+\left[\left(\frac{\partial^2\psi}{\partial x_1\partial x_2}\right)^2+\left(\frac{\partial^2\psi}{\partial x_2^2}-\frac{\partial^2\psi}{\partial x_1^2}\right)^2\right]^{\frac{k-1}{2}}\left[\left(\frac{\partial^2\psi}{\partial x_2^2}-\frac{\partial^2\psi}{\partial x_1^2}\right)\frac{\partial\psi}{\partial x_2}+\frac{\partial^2\psi}{\partial x_1\partial x_2}\frac{\partial\psi}{\partial x_1}\right]\cos\alpha_2\Bigg\}ds$$

$$-\oint\left\{\frac{p}{\rho}+\frac{1}{2}\left[\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2\right]\right\}\sqrt{\left(\frac{\partial\psi}{\partial x_1}\right)^2+\left(\frac{\partial\psi}{\partial x_2}\right)^2}.\cos\varphi\Bigg\}ds \quad (5\,,\,)$$

where

$$\nu^*=\frac{\mu^*}{\rho}=\gamma\varepsilon_0^{1-k} \tag{60}$$

$\alpha_1(x_1x_2)$, and $\varphi(x_1,x_2)$ are angles bet ween the unit normal **n** and the coordinate axes x_1,x_2, and the velocity vector v , respectively. 1 t is understood that these angles are known from the boundary conditions.

Let us assume that

$$\mathbf{v}.\mathbf{n}=0,\ \text{i.e.,}\cos\varphi=0,\quad \text{but } (\mathbf{v}\sigma).\mathbf{n}\neq 0 \tag{61}$$

which means that the external flow does not penetı ate the volume boundary S, and therefore, the exchange of energy between the volume v and the external flow is carried out by the viscous term ($\mathbf{v}\sigma$).

Then the last term in Eq. (59) vanishes.

Suppose that

$$\psi=\psi_1(t)\psi_2(x_1,x_2) \tag{62}$$

Then the Eq. (59 can be reduced to an ordinary differential equation for $\psi_1(t)$:

$$\dot{\psi}_1=\gamma^*(-A_1+A_2)\psi_1^k,\quad A_1>0, \tag{63}$$

where

$$A_1 = \frac{1}{2}\,\frac{\int_{\mathrm{v}} \left(\frac{\partial^2\psi_2}{\partial x_2} - \frac{\partial^2\psi_2}{\partial x_1}\right)^{k+1} dxdy}{\int_{\mathrm{v}} \left[\left(\frac{\partial\psi_2}{\partial x_1}\right)^2 + \left(\frac{\partial\psi_2}{\partial x_2}\right)^2\right] dxdy} = \text{const} \tag{64}$$

$$A_2 = \frac{2\int_{\mathrm{v}} a\left\{\left[\frac{\partial^2\psi_2}{\partial x_1\partial x_2} - \frac{\partial\psi_2}{\partial x_2} + \left(\frac{\partial^2\psi_2}{\partial x_1^2} - \frac{\partial^2\psi_2}{\partial x_2^2}\right)\frac{\partial\psi_2}{\partial x_1}\right]\cos\alpha_1\right.}{\int_{\mathrm{v}} \left[\left(\frac{\partial\psi_2}{\partial x_1}\right)^2 + \left(\frac{\partial\psi_2}{\partial x_2}\right)^2\right] dxdy}$$

$$+ \frac{\left.\int_{\mathrm{v}} \left[\frac{\partial^2\psi_2}{\partial x_1\partial x_2} - \frac{\partial\psi_2}{\partial x_1} + \left(\frac{\partial^2\psi_2}{\partial x_2^2} - \frac{\partial^2\psi_2}{\partial x_1^2}\right)\frac{\partial\psi_2}{\partial x_2}\right]\cos\alpha_2\right\} ds}{\int_{\mathrm{v}} \left[\left(\frac{\partial\psi_2}{\partial x_1}\right)^2 + \left(\frac{\partial\psi_2}{\partial x_2}\right)^2\right] dxdy} = \text{const} \tag{65}$$

and

$$a = \left[\left(\frac{\partial^2\psi_2}{\partial x_1\partial x_2}\right)^2 + \left(\frac{\partial^2\psi_2}{\partial x_2^2} - \frac{\partial^2\psi_2}{\partial x_1^2}\right)^2\right]^{\frac{k-1}{2}}. \tag{66}$$

We will analyze Eq. (52) for two cases when

$$A_2 - A_1 = -B^2 < 0 \tag{67}$$

and

$$A_2 - A_1 = B^2 > 0 \tag{68}$$

In the case *(67)* assume that

$$\psi_1(t=0) = \psi_1^o > 0 \tag{69}$$

which corresponds to the initial kinetic energy of the flow:

$$E_0 = \frac{\rho}{2}(\psi_1^0)^2 \int_v \left[\left(\frac{\partial \psi_2}{\partial x_1}\right)^2 + \left(\frac{\partial \psi_2}{\partial x_2}\right)^2\right] dx dy > 0 \tag{70}$$

Then Eq. (63) (under the condition (67)) describes the damping of the fluid motion due to viscous stress. It has regular solution

$$\psi_1 = [(\psi_1^0)^{1-k} - \nu^* B^2(1-k)t]^{\frac{1}{1-k}}, \tag{71}$$

and singular solution

$$\psi_1 \equiv 0 \tag{72}$$

solution.

In a finite time

$$t_0 = \frac{(\psi_1^0)^{1-k}}{\varepsilon_0^{1-k} \nu B^2(1 - k)} < \infty \text{ if } k < 1 \tag{73}$$

the regular solution (71) approaches equilibrium, i.e. the singular solution *(72).* This time depends upon the Constants k and ε_0 which can be found from experimental measurements of t_0.

It should be stressed that in the classical case, ($k = 1$), the solution to Eq. (63) approaches the equilibrium (72) asympt ot ically, i.e. $t_0 \to$ cm. That is why the parameter k found from Eq. (73) must be less than one.

In the Case (68) assume that

$$\psi_1(t = 0) = \psi_1^0 \to 0, \tag{74}$$

i.e., the liquid is in equilibrium,

$$E_0 = 0 \tag{75}$$

Under the condition (68), this equilibrium is unstable. Indeed, Eq. (68) subject to the initial rendition (74), has the form:

$$\psi_1 = \pm[\nu^* B^2(1-k)t]^{\frac{1}{1-k}} \tag{76}$$

The solution (76) possesses a remarkable property: it departs the equilibrium so fast that the velocity becomes finite despite vanishingly small disturbances (74) (compare with the classical case (3)). At the same time, with equal probability 1/2, this solution can become positive or negative which means that the solution att ains stochastic properties. It should be emphasized that this stochasticit y resul ts from the? relaxation of the Lipschitz condit icm at equilibria, and that, in turn, leads to failure of the? uniqueness of the solut ion. Thus, formal incorporation of an infinitesimal static friction in the constitutive equation of liquid allows one to explain the statistical nature of turbulence: in domains of supercritical Reynolds numbers, infinitesimal random components of the solution caused by the failure of the Lipschit z condition, are amplified by the mechanism of instability and lead to fully developed stochastic motions.

6. ATTRACTION TO EQUILIBRIUM AFTER SUDDEN MOVE OF BOUNDARIES

In the previous section we have discussed two fundament ally new properties of the mm- Lipschitzian model of liquid: a finite time of approaching equilibria, and occurrence of stochastic solutions to the modified Navier-Stokes equations. Both of these effects are in full agreement with experiments.

In this section we will illustrate? the modified model by example of an unsteady unidirectional flow induced by a sudden simult aneous move of bot h lower and upper boun daries.

Utilizing the constitutive law (36), one can write the following governing equation:

$$\frac{\partial u}{\partial t} = \nu^* (\frac{\partial u}{\partial y})^{k-1} \frac{\partial^2 u}{\partial y^2} \tag{77}$$

subject to the following boundary and initial conditions, respectively:

$$u(0,t) = u_0, \frac{\partial u}{\partial y}(\ell, t) = 0 \quad 0 < t < +\infty \tag{78}$$

$$u(y,0) = 0, \quad 0 < y < \ell \tag{79}$$

Here u is the flow velocity parallel to the horizontal axis x, y is the axis normal to the flow, 2ℓ is the distance between the lower and upper boundaries, and v* is the modified viscosity expressed by Eel. (60), u_o is the initial velocity of the boundaries, and k is expressed by Eq. (8).

The second boundary condition in (77) is formulated for the middle line between the boundaries in virtue of the symmetry of the problem.

For k = 1 one arrives at the classical diffusion equation:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \tag{80}$$

The solution to this equation subject to the boundary and initial conditions (78) and (79) is well known:

$$u(y,t) = u_0 - \frac{4u_0}{\pi} \sum_{n=0}^{+\infty} \frac{1}{2n+1} e^{-\frac{(2n^2+1)^2 \pi^2 \nu}{4\ell^2} t} \sin \frac{(2n+1)\pi}{2\ell} y \tag{81}$$

$$0 < y < \ell, \; 0 < t < +\infty \tag{82}$$

where ν is the kinematic viscosity.

obviously, this solution is valid for Eq. (77) in the domains where the condition (28) is satisfied, i.e., where

$$\left| \frac{\partial u}{\partial y} \right| \sim O(\varepsilon_0) \tag{83}$$

As follows from the solution (81), the condition (83) holds if

$$0 < t < t_*, \tag{84}$$

where

$$t_* \sim O(\frac{1}{\varepsilon_0}), \tag{85}$$

Turning to Eq. (77), let us introduce a new variable $\tilde{u}$:

$$u = \varepsilon_0 \tilde{u} \tag{86}$$

whose dimensionality is :

$$[\tilde{u}] = L \quad \text{since } [\varepsilon_0] = T^{-1} \tag{s7}$$

Therefore,

$$\left[\frac{\partial \tilde{u}}{\partial y}\right] \cdot \left[\left(\frac{\partial \tilde{u}}{\partial y}\right)^{k-1}\right] \tag{8s}$$

Then Eq. (77) can be rewritten in the form:

$$\frac{\partial \tilde{u}}{\partial t} = \nu \left(\frac{\partial \tilde{u}}{\partial y}\right)^{k-1} \frac{\partial^2 \tilde{u}}{\partial y^2} \tag{s9}$$

Since we are looking for the solution to (89) in the domain t > -t*, the boundary and initial conditions now are formulated as:

$$\tilde{u}(0, \cdot) = \tilde{u}_0, \quad \frac{\partial \tilde{u}}{\partial y} \ \ell, t) = 0, \quad t_* < t < +\infty \tag{90}$$

$$\tilde{u}(y, t_*) = \tilde{u}_*, \ 0 < y < \ell \tag{91}$$

Here

$$\tilde{u}_0 = \varepsilon_0 u_0, \; \tilde{u}_* = \varepsilon_0 u_* \tag{92}$$

where u^* is velocity at $t = t_*$ obtained from the classical solution (81) which is valid for

$$0 < t \leq t_* \tag{93}$$

Seeking the solution to Eq. (89) in the domain $t > t_*$ for $k < 1$ (see Eq. (8)) in the form

$$\tilde{u} = \tilde{u}_0 + u_1(t) u_2(y) \tag{94}$$

one obtains:

$$\dot{u}_1 + \lambda \nu u_1^k = 0, \; (\dot{u}_1 = du_1/dt), \lambda = \text{const} \tag{95}$$

$$u_2''(u_2')^{k-1} + \lambda \tilde{u}_2 = 0, \; (u_2' = du_2/dy) \tag{96}$$

the general solution to Eq. (96) has the form:

$$y = \frac{1}{k+1} \int \left(\frac{\lambda}{2} u_2^2 + C_1 \right)^{-\frac{1}{k+1}} du_2 + C_2 \tag{97}$$

where C_1 and C_2 are arbitrary const ants.

As follows from (97), y is a continuous function of k, so that

$$y(k) \to y(1) \text{ if } k \to 1, \; (\text{see Eq. (8)}) \tag{98}$$

Hence, Eq. (97) can be approximated by the classical solution:

$$y \simeq \frac{1}{2} \int \left(\frac{\lambda}{2} u_2^2 + C_1 \right)^{-\frac{1}{2}} du_2 + C_2 \tag{99}$$

and therefore,

$$u_2 = \frac{4}{\pi} \sum_{n=0}^{+\infty} \frac{1}{211 + 1} \sin \frac{2n+1}{2\ell} \pi y, \quad \lambda_n = \left[\frac{(2n+1)\pi}{2\ell} \right]^2, \qquad \text{(1oo)}$$

For each λ_n in (92), one can obtain a particular solution to Eq. (95):

$$u_1^{(n)} = [C_n - (1-k)\nu\lambda_n t]^{\frac{1}{1-k}}, C_n = \text{const} \tag{101}$$

However, since Eq. (95) is essentially nonlinear, the superposition principle is not applicable here. In order to circumvent this difficulty, we will confine ourselves to the solution for sufficiently large time.

$$t > t_* \quad (\text{see Eq. } (85)) \tag{102}$$

where the lowest mode corresponding to

$$\lambda_0 = \frac{\pi^2}{4\ell^2} \tag{103}$$

dominates over the others.

Then the solution to Eq. (89) reduces to:

$$\tilde{u} = \tilde{u}_0 \left\{ 1 - \frac{4}{\pi} \left[C_0 - (1-k)\frac{\pi^2 \nu}{4\ell^2} t \right]^{\frac{1}{1-k}} \right\} \sin \frac{\pi y}{2\ell}, \ t \geq t_* \tag{104}$$

The constant Co can be found by matching the solutions (81) and (104) at $t = t_*$, $y = \ell$:

$$C_0 = e^{-\frac{\pi^2(1-k)\nu}{4\ell^2\varepsilon_0}} + \frac{\pi^2(1-k)\nu}{4\ell^2\varepsilon_0} \tag{105}$$

while

$$\tilde{u}_* = u_0\left(1 - \frac{4}{\pi} e^{-\frac{\pi^2 \nu}{4\ell^2 \varepsilon}}\right)\sin\frac{\pi y}{2\ell} \tag{106}$$

Finally, the solution to Eq. (77) is:

$$u = u_0\left\{1 - \frac{4}{\pi}\left[e^{-\frac{\pi^2(1-k)\nu}{4\ell^2\varepsilon_0}} - (1-k)\frac{\pi^2(1-k)\nu}{4\ell^2}t\right]^{\frac{1}{1-k}}\sin\frac{\pi y}{2\ell},\ t \geq t_*\right. \tag{107}$$

while for $0 < t \leq t_*$ the solution can be presented in the classical form (81),

Although Eq. (107) represents an approximate solution to Eq. (77), it still preserves its fundamental property: the finite time t_0 of approaching the equilibrium:

$$t_* \ll t_0 = \frac{1}{\varepsilon_0} + \frac{4\ell^2}{(1-k)\pi^2\nu} e^{\frac{\pi^2(1-k)\nu}{4\ell^2\varepsilon_0}} < +\infty \tag{10s}$$

As could be expected, this time depends upon two new physical constants of the. liquid: k and ε_o.

7. SUDDEN START FROM REST

Continuing the analysis of the proposed model of a fluid, in this section we will pose " the following problem: find the velocit y field and the drag forces induced by a particle of a vanishingly small size suddenl y start ing from rest. This problem is very important in a variet y of physical contexts, such as the settling of sediment in a liquid, and the fall of mist droplets in air. Nevert heless, from a formal ma thereat i cal viewpoint, for a Newtonian liquid such a problem does not make much sense: all the hydrodynamical effects vanish when the size of the particle becomes infinitesimal.

Indeed, invoking the Stokes solution for a moving sphere,

$$\psi = ur^2 \sin^2\theta\left(\frac{3}{4}\frac{a}{r} - \frac{1}{4}\frac{a^3}{r^3}\right), \tag{109}$$

$$F = 6\pi a\mu u \tag{110}$$

one obtains:

$$\psi \rightarrow 0, \;\; F \rightarrow \;0 \;\; \text{if} \;\; a \rightarrow 0 \tag{111}$$

where a is the radius of the sphere.

But if this sphere moves in an unbounded volume, any finite size is "vanishingly small". That is why the smallness of the size? of a particle is actually understood as the smallness of the Reynolds number Re. However, expressing t 1 ıe drag force in (11 O) via the Reynolds number, one arrives at a singularity y for the drag coefficient:

$$C_D \cong \frac{24}{Re} \rightarrow \infty \;\; \text{if} \;\; Re \rightarrow 0 \tag{112}$$

Thus, the classical approach to the problem posed above gives only qualitative rather that quantitative results.

We will start with the plane flow in the domain of small velocity gradients where

$$\frac{\sqrt{I_2}}{\varepsilon_0} \ll 1 \tag{113}$$

Therefore, the momentum equations for this case can be reduced to the form (55), i.e.,

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 \psi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial x_2^2}\right) = \frac{\partial^2 \sigma_{11}}{\partial x_1 \partial x_2} + \frac{\partial^2 \sigma_{12}}{\partial x_2^2} - \frac{\partial^2 \sigma_{12}}{\partial x_1^2} - \frac{\partial^2 \sigma_{22}}{\partial x_1 \partial x_2} \tag{114}$$

in which σ_{11}, σ_{12}, and σ_{22} are expressed by Eqs. (32), *(33)* and *(34)*, respectively.

We will show that this equation has a class of solutions which is fundamentally different from those in the? classical case. For this purpose, let us seek the solution in the following form:

$$\psi = \alpha t^q (x_1^m + x_2^m) \tag{115}$$

Substituting (115) into Eq. (1 14), one obtains:

$$q = \frac{1}{1-k}, \; m = -\frac{2k}{1-k}, \; \alpha = \pm(1-k)^{\frac{k}{k-1}}(k-1)^{\frac{2k}{k-1}}\nu_* \tag{116}$$

The two signs for α can be expected if one recalls that, as follows from Eq. (8):

$$k = \frac{2n - 1}{2n+1} \tag{117}$$

where n is the one of the natural numbers

$$n = 1, 2, \ldots \text{ etc.} \tag{118}$$

Then

$$\frac{1}{1-k} = \frac{2n+1}{2}, \tag{119}$$

and the power $1/(1-k)$ includes the square root operation. (The expression in the square brackets in (116) is positive for k given by Eq. (117)).

The solution (1 15) in terms of velocities can be presented in the form:

$$v_1 = \pm x_2 \varepsilon_0 \left(\beta \frac{\gamma t}{x_2^2}\right)^{\frac{1}{1-k}}, \; v_2 = \pm x_1 \varepsilon \left(\beta \frac{\gamma t}{x_1^2}\right)^{\frac{1}{1-k}} \tag{120}$$

here

$$\beta = \pm(1+k)^k \; (k-1)^{k+1} = \text{const} \tag{121}$$

while the physical constant ε_0 is introduced by the constitutive equation (25).

One can verify that the expressions in the first brackets in Eq. (S7) have the dimension of velocity, and the expressions in the second brackets are dimensionless.

Substituting (87) into the momentum equations:

$$\frac{\partial v_1}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x_1} + \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2}, \frac{\partial v_2}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x_2} + \frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} \tag{122}$$

one concludes that

$$\sigma_{11} = 0, \quad \sigma_{22} = 0, \; \nabla p = 0, \text{ i.e., } p = \text{const} \tag{123}$$

This means that the velocit y field represents a shear flow.

We will start with the formal analysis of the solution (120), First it should be noticed that

$$v_1 = v_1^0 \pm \; x_2 \varepsilon_0 \left(\beta \frac{\gamma t}{x_2^2}\right)^{\frac{1}{1-k}}, v_2 = v_2^0 \pm \; x_1 \varepsilon_0 \left(\beta \frac{\gamma t}{x_1^2}\right)^{\frac{1}{1-k}} \tag{124}$$

are also the solutions to Eq. (1 14), or Eq. (122), and therefore, Eqs. (120) represent particular solutions to (122) subject to the initial conditions:

$$v_1 = 0, \quad v_2 = 0, \quad \text{at} \quad t = 0 \tag{125}$$

However, in addition to that, Eqs. (122) have a singular solution for the same initial conditions (125):

$$v_1 \equiv 0, \; v_2 \equiv 0 \tag{126}$$

which is not included in the family of the solutions (124). Obviously such a non-uniqueness of solution is a result of violation of the Lipschitz condition at the (equilibrium. As in cases analyzed in the previous sections (see Eq. (17) and (71)), the solution (126) is unstable: infinitesimal initial velocities

$$\pm v_1^0 \to 0, \quad \pm v_2^0 \to 0 \quad \text{at} \quad t = 0 \tag{127}$$

transfer it into one of the solutions (120) which will rapidly escape from the equilibrium. It is important to emphasize that the signs of the solutions (1'20) are defined by the signs of the init ial conditions (127) which are random. Actually this is the origin of stochastisity of the solutions to the Navier-Stokes equations modified to tile form (39).

However, one should recall that the solut icm (120) is vali d only for those domains where the condition (2S) is still true, i.e., when

$$\left| \frac{\partial v_1}{\partial x_2} \right|, \left| \frac{\partial v_2}{\partial x_1} \right| \sim O(\varepsilon_0) \tag{128}$$

in which ε_o is defined by Eq. (25).

As follows from Eq. (120):

$$\left| \frac{\partial v_1}{\partial x_2} \right| = \frac{k}{1-k} \beta \varepsilon_0 \left(\frac{\gamma t}{x_2^2} \right)^{\frac{1}{1-k}}, \quad \left| \frac{\partial v_2}{\partial x_1} \right| = \frac{k}{1-k} \beta \varepsilon_0 \left(\frac{\gamma t}{x_1^2} \right)^{\frac{1}{1-k}} \tag{129}$$

Hence, the solution (120) is not valid for the domain

$$x_1^2, \ x_2^2 \ > \ r_0^2 \tag{130}$$

while

$$r_0^2 \cong O\left(\frac{\nu t}{\gamma} \right), \ \gamma = \left(\frac{1-k}{k\beta} \right)^{1-k} = \text{const} \tag{131}$$

For this domain one has to apply the original version of the momentum equations (39) which include the convective components of the acceleration (being dropped in Eq. (144)).

Let us now concentrate on some physical effects described by the solution (120). Consider a rigid particle at rest. Then it must be that:

$$v_1 = v_2 \equiv 0, \ \ x_1^2, x_2^2 \ < \ r_0^2 \tag{132}$$

This condition can be satisfied if one? combines the positive and the negative branches of the solution (120) as following:

$$v_1 = v_1' + v_1'', \quad v_2 = v_2' + v_2'' \tag{133}$$

where

$$v_1' = \begin{cases} +\beta(x_2\varepsilon_0)(\frac{\gamma t}{x_2^2})^{\frac{1}{1-k}} & \text{for } x \geq --r_0 \\ 0 & \text{otherwise} \end{cases} \tag{134}$$

$$v_1'' = \begin{cases} -\beta(x_2\varepsilon_0)(\frac{\gamma t}{x_2^2})^{\frac{1}{1-k}} & \text{for } x_2 \geq r_0 \\ 0 & \text{otherwise} \end{cases} \tag{135}$$

$$v_2' = \begin{cases} +\beta(x_1\varepsilon_0)(\frac{\gamma t}{x_1^2})^{\frac{1}{1-k}} & \text{for } x_1 \geq -r_0 \\ 0 & \text{otherwise} \end{cases} \tag{136}$$

$$v_3' = \begin{cases} -\beta(x_1\varepsilon_0)(\frac{\gamma t}{x_1^2})^{\frac{1}{1-k}} & \text{for } x_1 \geq r_0 \\ 0 & \text{otherwise} \end{cases} \tag{137}$$

Thus, the solution (134)-(137) describes the flow around a rigid particle of the radius $r_* = r_o$ at rest. But before discussing the cause of this flow, let us find the force of interaction between the flow and the particle.

Obviously, this force can be found as:

$$F_f = \int_0^{2\pi} \sigma_{12}\ r^* d\varphi, \text{ i.e.,} \tag{138}$$

$$F_f = 4\pi r_* \rho\nu\varepsilon_0 \left(\frac{k\beta}{1-k}\right)^k \left(\frac{\nu t}{r_*^2}\right)^{\frac{k}{1-k}} \tag{139}$$

This equation is valid only until the condition (128) is satisfied, i.e. (with reference to Eq. (131)) until

$$F_f \leq F_{f\max} \cong O\,(4\pi\rho\nu\varepsilon_0) \tag{140}$$

Let us assume that some external force? F is applied to the particle at rest. In contradistinction to the classical case, the flow starts moving, first raising the reaction force (139), and only after that the particle starts moving. This situation resembles the behavior of a rigid body on a rough surface which can start moving only after the applied force exceeds the maximum static friction.

The maximum force due to infinitesimal static friction of the liquid is evaluated by Eq. (140). It depends upon ε_o which is the physical constant characterizing this liquid.

The tim(?-delay for the? motion of the particle is found from the? condition (131)

$$\Delta t \sim O\,(\frac{\gamma r_*^2}{\nu}) \tag{141}$$

where ν is given by Eq. (131).

As follows from Eq. (141), this delay depends upon k which is another physical constant characterizing the liquid.

For

$$t > \Delta t, \text{or } F > F_{f\max}, \tag{142}$$

the? particle starts moving, and one has to apply the original version of the momentum equations (39) which in this domain will mine.idt? with the Navier-Stokes equations. This means that in the domain (142) the velocity field and the drag force can be found from the Stokes formulas (109) and (112).

8. PHENOMENOLOGICAL APPROACH

As emphasized in the previous sections, both constants k and ε_o (see Eqs. (8) and (26)) represent additional physical properties of liquids, and therefore, they must be found from experiments. However, in this section we will find both k and ε_o based upon phenomenological concepts. For that purpose let us compare the solutions to Eqs. (77) and (80) for

$$t \geq \frac{1}{\varepsilon_0} = t_* \tag{143}$$

The solution to Eq. (77) expressed by Eq. (107) approaches the equilibrium at $t = t_0$ where t_0 is defined by Eq. (108), The solution to Eq. (SO) expressed by Eq. (81), theoretically never approaches the equilibrium; however, in finite time it approaches a domain of insensitivity where the velocity u_{00} is so small that it cannot be detected by sensors. Hence, the? actual time? t'_0 of approaching the equilibriun by the solution (81) can be found as:

$$t'_0 = \frac{1}{\varepsilon_0} - \frac{4\ell}{\pi^2 \nu} \ell n \left[\frac{\pi}{4} \left(1 - \xi \right) \right] \tag{144}$$

where

$$\xi = \frac{u_{00}}{u_*} \ll 1 \tag{145}$$

in which u_{00} is the value of the smallest detectable velocity, and

$$u_* = u \quad \text{at} \quad t = t_* \tag{146}$$

Based upon up-t o-date level of measurement t echniques,

$$\xi \cong 0.01. \tag{147}$$

Recalling that the solutions (S1) and (107) are different only within a small neighborhood of the equilibrium, let us find "equivalent" values of k = k* and $\varepsilon_0 = \varepsilon_0^*$ from the Condition that the time t_0 of approaching the equilibrium by the solution (107), and the time t'_0 of approaching the domain of insensitivity are the same. Equating t_0 and t'_0 from Eqs. (108), and (144), respectively, one arrives at a phenomenological relationship between k* and ε_0^*:

$$\varepsilon_0^* = -\frac{\pi^2 (1 - k^*) \nu}{4\ell^2 \ell n\{(k^* - 1) \ell n[\frac{\pi}{4}(1 - \xi)]\}}, \quad \xi \cong 0.01 \tag{148}$$

The second relationship between these parameters can be derived from the following phenomenological concept. Turning to the constitutive law (26), one has to provide the property that this law is sufficiently dose to the linear one for large velocity gradients (which, however, are smaller than the molecular velocit y gradients). Since the molecular velocit y gradients are of the order of 1/~ where τ is the Maxwell relaxation time, one can write:

$$\left(\frac{1}{\varepsilon_0^* \tau}\right)^{k-1} \cong 1 - \xi \tag{149}$$

This condition guarantees that the difference between the constitutive law (26) and the linear law are within the bounds of accuracy of the velocity measurements for the entire domain where the equations for a Newtonian liquid are applicable.

As follows from Eq. (149):

$$\varepsilon_0^* = \frac{1}{\tau}(1 - \xi)^{\frac{1}{1-k_*}} \tag{150}$$

Eels. (148) and (150) express the phenomenological versions of the physical parameters k and ε_o via the physical const ant τ and the accuracy of measurements ξ. In contrast to k and .5$_0$, the const ants k* and ε_o^* are problem-dep endent. Indeed, Eq. (148) was derived from the solution to a particular problem discussed in the section 6, and that is wily it contains the representative length ℓ. However, despite this limitation, Eqs. (148) and (150) provide a good evaluation of the order of these parameters.

Eqs. (148) and (1 50) solved for the water at 200C become:

$$\tau = 10^{-9}\text{sec}, \ \xi = 0.01, \ \ \nu = 10^{-2}\frac{\text{cm}^2}{\text{sec}}, \ \ \ell = 10 \text{ cm} \tag{151}$$

and lead to the following values of k^* and ε_0^* :

$$k = \frac{3109}{3111}, \quad \varepsilon_0 = 2.66.10^{-5}\text{scc} \tag{152}$$

Now we can evaluate the parameters β, γ, r_0 , $F_{\max}$ and Δt introduced in the previous section (see Eqs. (121),(131), (140) and (141), respectively:

$$\beta \cong 2.10^{-7}, \quad \gamma \cong 1, \quad r_0 \cong \gamma t, \quad F_{\max} \cong 4.266\ \pi 10^{-5} \rho\nu r_* \tag{153}$$

$$\Delta t \simeq \frac{r_*^2}{\nu}$$

9. APPLICATION TO ACOUSTICS

So far we have considered only incompressible fluids, However, all the modifications of the constitutive law can be applied to gases too. Indeed, turning to Eq. (25), one can write [5]:

$$\rho\mathcal{D} = 2\mu'\left(\frac{\sqrt{I_2}}{\varepsilon_0}\right)^{k-1} + \left(\mu' + \frac{1}{2}\mu''\right)\left(\frac{I_1}{\varepsilon_0}\right)^{k-1} \tag{154}$$

where μ'' is the second viscosity. Then, instead of Eq. (25) one obtains:

$$\sigma_{jk} = 2\mu'\left(\frac{\sqrt{I_2}}{\varepsilon_0}\right)^{k-1}\varepsilon_{jk} + \left[\mu''\left(\frac{I_1}{\varepsilon_0}\right)^{k-1}\varepsilon_{\ell\ell} - p\right]\delta_{jk} \tag{155}$$

Here

$$I_1 = \varepsilon_{ii}, \tag{156}$$

is the first invariant of the rat e-of-st rain tensor, and δ_{jk} is the Kronecker's delta symbol.

The. momentum and mass conservation actuations instead of (39), and (40), take the following form, respectively:

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\nabla\mathbf{v}\right) = -\nabla p + \mu'\left\{\left(\frac{\sqrt{I_2}}{\varepsilon_0}\right)^{k-1}\left[\nabla\cdot(\nabla\mathbf{v} + \nabla\mathbf{v}^T)\right]\right.$$
$$\left. + (\nabla\mathbf{v}\cdot\ \nabla\mathbf{v}^T).\nabla\left(\frac{\sqrt{I_2}}{\varepsilon_0}\right)^{k-1}\right\} + (\mu' + \frac{1}{3}\mu'')\left[\left(\frac{I_2}{\varepsilon_0}\right)^{k-1}\nabla\nabla.\mathbf{v} + \nabla.\mathbf{v}\nabla\left(\frac{I_2}{\varepsilon_0}\right)^{k-1}\right] \tag{157}$$

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho\mathbf{v}) = 0. \tag{15s}$$

For a one-dimensional compressible viscous flow, the normal stress is given by the following Expression:

$$\sigma_{xx} = -p + \left(\frac{4}{3}\mu'' + \mu'\right)\varepsilon_0^{1-k}\left(\frac{\partial u}{\partial x}\right)^k \tag{159}$$

while the momentum and mass conservation equations describing acoustic waves reduce to:

$$\rho_0 \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \tilde{\mu}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)^k \tag{160}$$

$$\frac{\partial \rho}{\partial t} = -\rho_0 \frac{\partial u}{\partial x}, \tag{161}$$

in which

$$\tilde{\mu} = \left(\frac{4}{3}\mu'' + \mu'\right)\varepsilon_0^{1-k} \tag{162}$$

and ρ_0 is the unperturbed value of the density.

After elimination of the pressure p, one arrives at the governing equation for acoustic waves :

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} + \tilde{\nu}\frac{\partial^2}{\partial t \partial x}\left(\frac{\partial u}{\partial x}\right)^k, \quad c^2 = \frac{dp}{d\rho}, \tilde{\nu} = \frac{\tilde{\mu}}{\rho} \tag{163}$$

Usually the last term describing viscous effects is ignored since it is much smaller than the elastic term. However, in our case the viscous term can be as sizable as the elastic term if the velocity gradients are small, i.e., if

$$\frac{\partial u}{\partial x} < \varepsilon_0 \tag{164}$$

In order to simplify Eq. (163) let us introduce. a new variable τ instead of t:

$$\tau = t - \frac{x}{c} \tag{165}$$

Then Eq. (163) reduces to:

$$\frac{\partial u}{\partial x} = a^2 \frac{\partial}{\partial \tau}\left(\frac{\partial u}{\partial \tau}\right)^k, \quad a^2 = \frac{\nu}{c^3}(\varepsilon_0 c)^{1-k} \tag{166}$$

Eq. (166) is identical to Eq. (77) if the variables x and τ are replaced by t and y, respectively.

Let us find an approximate solution to Eq. (166) subject to the following initial condition:

$$u(\mathbf{T}) = \varphi(\tau), \; -\infty < x < +\infty \tag{167}$$

Seeking the solution in the form:

$$u = \int u_\lambda(x) f_\lambda(\tau) d\lambda \tag{168}$$

and substituting (168) into (166) one obtains:

$$f_\lambda = [(f'_\lambda)^k 1' \tag{169}$$

These equations are similar to Eqs. (96), and therefore, their solutions (97) can be simplified to the approximations (99) which lead to the following form:

$$f''_\lambda + \lambda^2 f_\lambda = 0, \quad f_\lambda = e^{i\lambda\tau}. \tag{170}$$

Then the functions u_λ can be found from the following equations:

$$\frac{du_\lambda}{dx} + a^2 \lambda^2 x^k = 0 \tag{171}$$

whence

$$u_\lambda = u_\lambda^0 \left[1 - \frac{(1-k)a^2\lambda^2}{u_\lambda^0} x\right]^{\frac{1}{1-k}} \tag{172}$$

Substituting (170) and (172) into (168), one arrives at a Fourier integral:

$$u = \int u_\lambda^0 \left[1 - \frac{(1-k)a^2\lambda^2}{u_\lambda^0} x\right]^{\frac{1}{1-k}} e^{i\lambda\tau} d\lambda \tag{173}$$

Since

$$\varphi(\tau) = \int u_\lambda^0 \, e^{i\lambda\tau} d\lambda, \tag{174}$$

the values of u_λ^0 can be found as a Fourier coefficients for $\varphi(\tau)$:

$$u_\lambda^0 = \frac{1}{2\pi} \int \varphi(\xi) \, e^{-i\lambda\tau} \, d\xi \tag{175}$$

Substituting (175) into Eq. (168) one obtains the solution in the following integral form:

$$u = \int_\infty^\infty \left\{ \left[\frac{1}{2\pi} \int_\xi \varphi(\xi) \, e^{-i\lambda\tau} \, d\xi\right]^{1-k} - (1-k)a^2\lambda^2 x \right\}^{\frac{1}{1-k}} e^{i\lambda\tau} \, d\lambda \tag{176}$$

Let us assume that

$$u(O,\ \tau) = \varphi(\tau) = u_0 \sin\omega\tau \tag{177}$$

Then

$$\lambda^2 = \omega^{k+1},\ u_\lambda^0 = u_0 \tag{178}$$

and, as follows from Eq. (1 76)

$$u = \left[u_0^{1-k} - (1-k)a^2\omega^{k+1}x\right]^{\frac{1}{1-k}} \sin\omega(t - \frac{x}{c}) \tag{179}$$

Eq. (179) describes a travelling acoustic wave generated by a sinusoidal source of sound located at $x = 0$. The wave propagates with the classical acoustic speed $c^2 = dp/d\rho$, but the amplitude of this wave gradually decreases with increase of the dist ante x from the source.

Eventually, at the distance

$$|x| \geq \ell \frac{u_0^{1-k}\omega^{(1-k)}}{(1-k)a^2\omega^2} = \frac{(1-k)c^3}{\nu\omega^2}\left(\frac{u_0\omega}{c\varepsilon_0}\right)^{1-k} \tag{180}$$

the sound vanishes being absorbed due? to the viscosity.

As follows from Eq. (180), the critical distance ℓ depends upon two new physi cal const ants: k and ε_0. As can be expected, this distance decreases with the increase of viscosity ν and the frequency ω.

It should be recalled that in the classical case the solution to the same problem, instead of (1 79), would be:

$$u = u_0\, e^{-a^2\omega^2 x} \sin\omega(t - \frac{x}{c}) \tag{181}$$

This means that an acoustic wave is never fully absorbed; the distance over which the source of sound can be detected, is infinitely large. Eq. (180) gives a correction to this idealized result stating that this distance is finite.

10. APPLICATION TO ELASTIC BODIES

Effects, similar to those described in viscous fluids, occur in elastic bodies if dissipation processes are. taken into account. Indeed, in this case the total stress tensor can be combined from the elastic and viscous components:

$$\sigma = \sigma_e + \sigma_v, \tag{182}$$

while usually

$$\sigma_e >> \sigma_v \tag{183}$$

However, in domains close to equilibria, the elastic stresses vanish, and

$$\sigma_v \gg \sigma_e \tag{184}$$

But in these domains the. viscous stress can be represented by Eq. (26), and therefore, the governing equations take the form (55). This means that all the effects described above including finite time of approaching equilibria, non-uniqueness of solutions starting from equilibria, as well as finite distance of absorption of an acoustic wave, occur in elastic bodies in the domains close to equilibria.

11. DISCUSSION AND CONCLUSION

Plasticity effects are well-pronounced in heavy viscous liquids such as lubricators and dyes, but they have never been studied in classical Newtonian fluids like water, or air, presumably because they were expected to be vanishingly small. Our analysis demonstrates that although the quantitative contribution of these effects is small indeed, qualitatively even infinitesimal static friction leads to two new fundamental properties which are: the theoretically finite times of approaching equilibria, and the non-uniqueness of solutions which start at equilibria.

The first property can be associated with the paradox of irreversibility - one of the most fundamental and yet not fully understood problems in physics. Indeed, the concept of viscosity can be derived, on the microscopic level of description, from the fully reversible equations of Hamiltonian dynamics. This means that the irreversible processes in viscous flows are completely composed of reversible events. One of the most convincing and well accepted explanations of this paradox is that the change from an ordered arrangement to a disordered arrangement on the microscopic level as a source of irreversibility is much more probable than a change in the opposite direction.

In other words, any macroscopic system, in principle, can return to its initial state passing through all of its previous states; however, the probability of such an event is so small (but not zero!), that the period of time during which this event can occur is extremely large in comparison to the time scale of the macroscopic motions. However, the

Navier-Stokes equations, or their simplified version - the diffusion equation - do not have any time sale: the time of approaching an equilibrium is unbounded, and therefore, these equations exclude any reversible solutions even if $t \to \infty$. The only logical way out of this situation is to introduce a time-scale into the Navier-Stokes equations so that the time of approaching an equilibrium will be finite. Then one can argue that this time is not large enough to include reversible solutions. Actually this time scale was introduced by relaxing the Lipschitz condition at the equilibrium states (see Eq. 52). Simple experiments which allow one to find the constants defining this scale were also described.

The second property is linked to another fundamental, but still unsolved problem of theoretical physics - the problem of turbulence. From a formal mathematical viewpoint, turbulence results from dynamical instability of the Navier-Stokes equations when the Reynolds number exceeds certain critical values, and it is described by stochastic. solutions. But how can such solutions occur from a fully deterministic model? A physical explanation is very simple: possible uncertainties and small errors (which always can be interpreted as random components of initial conditions) are amplified by the mechanism of instability, and that leads to the stochasticity of the solutions for supercritical Reynolds numbers. In other words, turbulence is caused by a random input into the fully deterministic Navier-Stokes equations. However, a mathematician can argue that, in principle, there is always a possibility that there are no uncertainties or errors in initial conditions at all, and then the solutions will never become stochastic. Tile modified version of the Navier-Stokes equations attains a very fundamental new property: it generates stochasticity as a result of the non-uniqueness of the solution which, in turn, follows from relaxation of the Lipschitz condition at equilibrium states. In cases of dynamical instability the random components of the solution are amplified and that leads to stochastic solutions simulating turbulence. Otherwise these random components decay and vanish.

It should be stressed that although the qualitative differences between the. classical and modified Navier- Stokes equations are fundamental, all the new effects emerge within vanishingly small neighborhoods of equilibrium states which are the only domains where the modified governing equations are different from classical. This means that the formal differences between the solutions to classical and modified models may be not detectable in domains which do not include equilibrium states.

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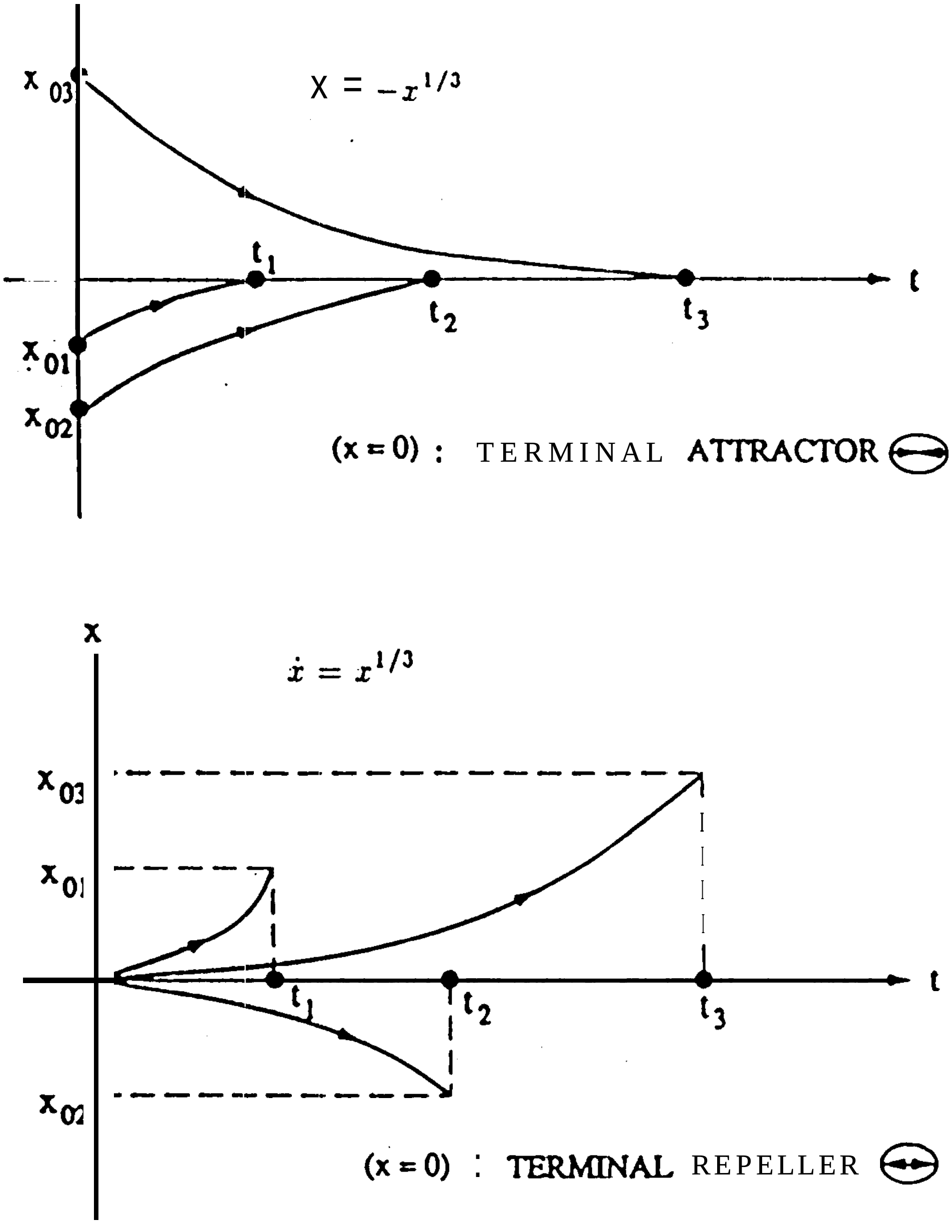

Figure 1.

Behavior of terminal attractor and repeller.

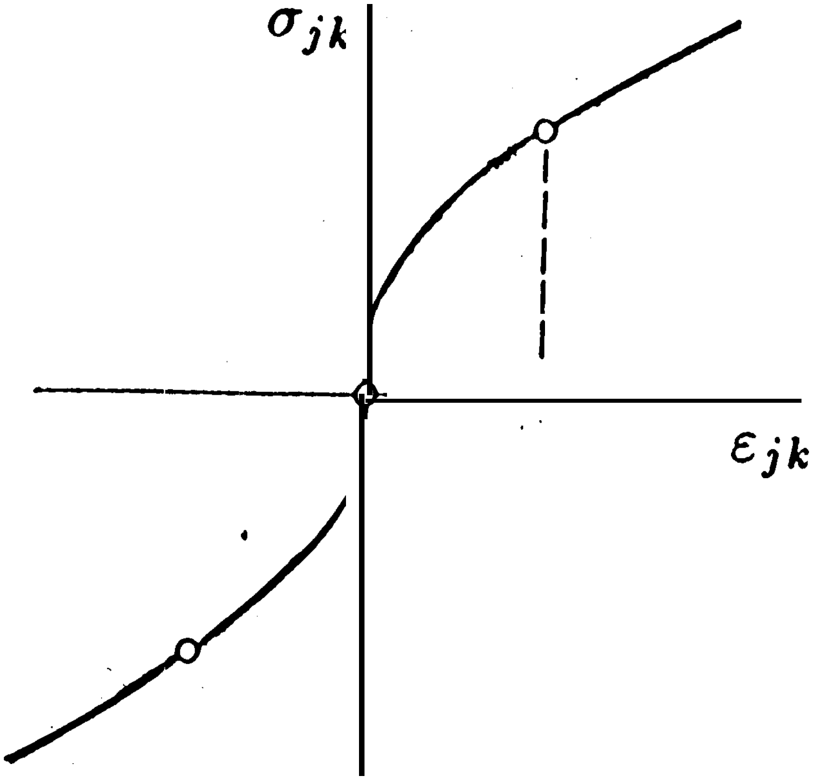

Figure 2

Limit case of stress-strain relationship for viscoplastic body.

APPENDIX

APPENDIX

The governing equations of classical dynamics based upon the? Newton laws:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} - \frac{\partial R}{\partial \dot{q}_i}, \; i = 1, 2, \cdots n \tag{1}$$

where L is the Lagrangian, and q, $\dot{q}_i$ are the general zed coordinates and velocities, include a dissipation function $R(\dot{q}_i \dot{q}_j)$ which is associated with the friction forces:

$$F_i(\dot{q}_1, \dot{q}_2, \cdots \dot{q}_n) = -\frac{\partial R}{\partial \dot{q}_i} \tag{2}$$

The structure of the functions (2) does not follow from the Newton's laws, and, strictly speaking, some additional assumptions should be made in order to define it. The "natural" assumption (which has been never challenged) is that these functions can be expanded in Taylor series with respect to an equilibrium state

$$\dot{q}_i = 0 \tag{3}$$

Obviously this requires the existence of the derivatives:

$$\left| \frac{\partial F_i}{\partial \dot{q}_j} \right| < \infty \text{ at } \dot{q}_j \to 0 \tag{4}$$

i.e., F_i must satisfy the Lipschitz condition. This condition allows one to describe the Newtonian dynamics within the mathematical framework of c.lassie.al theory of differential equations. However, there is a certain price paid for such a mathematical " convenience": the Newtonian dynamics with dissipative forces remains fully reversible in the sense that its governing equations are invariant with respect to time inversion, $t + -t$. As stressed by Prigogine, I., in this view future and past play the same role: nothing can appear in future which could not already exist in past since the trajectories followed by particles can never cross (unless i? $\to \pm\infty$). This means that classical dynamics cannot explain the emergency of new dynamical patterns in nature in the. same way in which non-equilibrium thermodynamics does.

In order to trivialize the mathematical part of our argumentation let us consider an one-dimensional motion of a particle decelerated by a friction force:

$$m\dot{v} = F(v) \tag{5}$$

in which m is mass, and v is velocity. Invoking the assumption (4) one can linearize the force F with respect to the equilibrium $v = 0$:

$$F \to -\alpha v \text{ at } v \to 0, \ \alpha = -\left(\frac{\partial F}{\partial v}\right)_{v=o} > 0 \tag{6}$$

and the solution to (5) for $v \to 0$ is:

$$v = v_o e^{-\frac{\alpha}{m}t} \to 0 \text{ at } t \to \infty, \ v_o = v(0) \tag{7}$$

As follows from (7), the equilibrium $v = 0$ cannot be approached in finite time. The usual explanation of such a paradox is that,, to accuracy of our limited scale of observation, the particle "actually" approaches the equilibrium in finite? time. In other words, eventually the trajectories (7) and $v = 0$ become so close that we cannot distinguish them. The same type of explanation is used for the emergence of chaos: if two trajectories originally are "very dose", and then they diverge exponentially, the same initial conditions can be applied to either of them, and therefore, the. motion cannot be traced.

Hence, there are variety of phenomena whose explanations cannot be based directly upon the classical dynamics: in addition, they require some "words" about a scale of observation, '(viny close" trajectories, etc..

In this note we propose a new structure of the. dissipation forces which eliminates the paradox discussed above and makes the Newtonian dynamics irreversible. The main properties of the new structure are based upon a violation of the Lipschitz condition (4). Turning to the example (5), let us assume that

$$F = -\alpha v^k, \ k = \frac{p}{p+2} < 1 \tag{8}$$

in which p is an odd number.

By selecting large p, one can make? k close to 1 so that Eqs. (6) and (8) will be almost identical everywhere excluding a small neighborhood of the equilibrium point $v = 0$, while, as follows from (8), at this point:

$$\frac{\partial F}{\partial v} = -k\alpha v^{k-1} \to \infty \text{ at } v \to 0 \tag{9}$$

Hence, the condition (4) is violated, the friction force grows sharply at the equilibrium point, and then it gradually approaches the straight line (6). This effect can be interpreted as a jump from static to kinetic friction.

It appears that this '(small" difference between the friction forces (6) and (S) leads to fundamental changes in Newtonian dynamics.

Firstly, the time of approaching the equilibrium $v = O$ becomes finite. Indeed, as follows from Eqs. (5) and (8):

$$t_o = -\int_{v_o}^{0} \frac{mdv}{\alpha v^k} = \frac{mv_o^{1-k}}{\alpha(1-k)} < \infty \tag{1o}$$

Obviously this integral diverges in the classical case when $k \geq 1$.

Secondly, the motion described by Eq. (8)

$$v = [v_o^{1-k} - \frac{\alpha}{m}(1-k)t]^{\frac{1}{1-k}} \tag{11}$$

is irreversible since the time-backward motion

$$v_- = \{[v_o^{1-k} - \frac{\alpha}{m}(1 - k)(-t)]^{q+2}\}^{1/2} \tag{12}$$

is imaginary. (one can verify that the classical version of this mot ion (7) is fully reversible if t < m) .

As shown by Zak, M. [2], the equilibrium point v = O of Eq. (8) represents a terminal attractor which is "infinitely' stable and is intersected by all the attracted transients, Fig. 2. Therefore, the uniqueness of the solution at v = O is violated, and the motion for $t < t_o$ (see Eel. (10)) is totally "forgotten". (This is a mathematical implication of irreversibility of the dynamics (8)).

So far we were cone.emecl with stabilizing effects of dissipative forces. However, as wall-known from dynamics of non-conservative systems, these forces can destabilize the mot ion when they feed the external energy into the system (t he transmission of energy from laminar to turbulent flow in fluid dynamics, or from rotations to oscillations in dynamics of flexible s yst ems). In order to capt ure the fundament al propert ies of these effects in case of "terminal" dissipative force? (8) by using the simplest mathematical model, let us turn to Eel. (5) and assume that now the friction force feeds energy into the system:

$$m\dot{v} = \alpha v^k, \ k = \frac{p}{p+2} < 1 \tag{13}$$

One can verify that for Eq. (13) the, (equilibrium point v = O becomes a terminal repeller, and since

$$\frac{d\dot{v}}{dv} = k\alpha v^{k-1} \to \infty \text{ at } v \to 0 \tag{14}$$

it is "infinitely' unstable. If the initial condition is infinitely dose to this repeller, the transient solution will escape it during a finite time period:

$$t_o = \int_{\varepsilon \to 0}^{“0} \frac{dv}{v^k} = \frac{v_o^{1-k}}{1-k} < \infty \tag{15}$$

while for a regular repeller, the time would be infini te.

As in the case of a terminal attractor, here the motion is also irreversible since the inversion of time in the solution to Eq. (14).

$$v = \pm[\frac{\alpha}{m}(1-k)t]^{\frac{1}{1-k}} \tag{16}$$

leads to imaginary values of v.

But in addition to that, terminal repellers possess even more surprising characteristics: the solution (16) becomes tot ally unpredict able. Indeed, t wo different motions described by the solution (16) are possible for "almost the same" ($v_o = +\varepsilon + O$, or $v_o = -\varepsilon \to O$ at $t = \to O$) initial conditions. The most essent ial propert y of this result is that the divergence of these two solutions is characterized by an unboundedrate:

$$\sigma = \lim_{t \to t_o} (\frac{1}{t} \ln \frac{t^{1/1-k}}{|v_o|}) \to \infty \text{ at } |v_o| \to 0 \tag{17}$$

In contrast to the classical case where $t_o \to \infty$, here o can be defined in an arbitrarily small time interval t_o, since dining this interval the initial infinitesimal distance bet ween the solutions becomes finite. Thus, a terminal repeller represents a vanishingly short, but infinitely powerful "pulse of unpredictability" which is pumped into the system via terminal dissipative forces. Obviously failure of the uniqueness of the solution here results from the violation of the Lipschitz rendition (4) at v = O.

As 'known from classical dynamics, the combination of stabilizing and destabilizing effects can lead to a new phenomenon: chaos. 1 n order to describe similar effects in dynamics with terminal dissipative forces let us slightly modify Eq. (13):

$$m\dot{v} = \alpha v^k \cos\ \omega t \tag{18}$$

Here stabilization and destabilization effects alternate. With the initial condition v --i 0 at t $\rightarrow$ O the exact solution to Eq. (18) consists of a regular solution:

$$v = \pm[\frac{\alpha(1-k)}{m\omega}\sin\omega t]^{\frac{1}{1-k}},\ v + O; \tag{19}$$

and a singular solution:

$$v = 0 \tag{20}$$

During the first period O < $t < \pi/2\omega$ the equilibrium point (20) is a terminal repeller. Therefore, within this period, the motion can follow one of t wo possible trajectories (19) (each with the probability 1/2) which diverge with unbounded rate (17) at v = O. During the next period $\pi/2\omega < t < 3\pi/2\omega$ the equilibrium point (20) becomes a terminal attractor; the solution approaches it at $t = \pi\omega$ and it remains motionless until $t > 3\pi/2\omega$. After that the terminal attractor converts into terminal repeller, and the solution escapes again, etc.

It is important to notice that each time the system escapes the terminal repeller, the solution splits into two symmetric branches, so that there is 2^n possible scenarios of the. oscillations with respect to the renter v = O, while each scenario has the probability 2^{-n} (n is the number of cycles). Hence, the motion (19) resembles chaotic oscillations known from classical dynamics: it combines random characterist ics wit] ı the attraction to a center. However, in classical case the chaos is caused by a supersensitivity to the initial conditions, while the uniqueness of the solution for fixed initial conditions is guaranteed. In contrast to that, the chaos in the oscillations (19) is caused by the failure of the uniqueness of the solution at the equilibrium points, and it has a well organized probabilistic structure. Since the time of approaching the equilibrium point v = O by the solution (19) is finite, this type of chaos can be died terminal.